INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids **31**: 299–308 (1999)

A HIGHER-ORDER COVOLUME METHOD FOR PLANAR DIV–CURL PROBLEMS

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SUMMARY

Covolume methods constitute a generalization to unstructured meshes of classical staggered mesh techniques. In this paper, a fourth-order method is proposed and it is proved rigorously that the order is indeed 4 in a standard norm. This result is for structured meshes only, and for div–curl equations in two-dimensional space. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: div–curl systems; covolume methods; higher-order methods; finite difference methods

1. INTRODUCTION

In this report, a higher-order covolume scheme for planar div–curl problems is constructed. Error estimates and a numerical example are given to show that the convergence rate of the covolume scheme is fourth-order in h, the uniform mesh size.

Div–curl systems appear in fluid dynamics [1,2], in electromagnetics [3,4] and many other applications. The first-order systems are often solved using indirect methods including potential formulations or Biot–Savart-type integrals [5] and least-squares [6]. Since potential formulations can have spurious mode problems [7] and the Biot–Savart approach needs special handling of boundaries a different treatment may be desirable. A direct discretization of planar div–curl problems was proposed by Nicolaides [8]. The scheme used a Delaunay–Voronoi mesh system to discretize the div equation in the primal cells and the curl equation in the dual cells. It was shown that the convergence rate of this complementary volume, or *covolume* for short, method is first-order for general unstructured meshes and second-order for a class of smoothly varying meshes.

Let Ω denote a rectangular domain in R^2 with boundary Γ and unit outward normal $\mathbf{n}$. If $\mathbf{u} = (u, v)$ denotes a vector field in R^2, a div–curl system is

$$\operatorname{div} \mathbf{u} = f \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{curl} \mathbf{u} = g \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} \cdot \mathbf{n} = l \quad \text{on } \Gamma, \tag{1.3}$$

where f and g are scalar functions defined in Ω, and l is a given function defined on Γ. It is assumed that

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CCC 0271–2091/99/170299–10$17.50

$$\int_{\Gamma} l \, ds = \int_{\Omega} f \, d\mathbf{x}. \tag{1.4}$$

It is known that if $f, g \in L^2(\Omega)$ and $l \in H^{1/2}(\Gamma)$, the system (1.1)–(1.3) has a unique solution $\mathbf{u} \in (H^1(\Omega))^2$ (see [9]).

2. MESH NOTATIONS AND COVOLUME METHODS

Consider a uniform grid with the grid size h on Ω and take this uniform grid as the primal mesh. A dual mesh is formed by connecting the centers of adjacent square cells and consists of staggered square cells. It follows that the primal edges are orthogonal to the corresponding dual edges. This reciprocal orthogonality is very important in the construction and analysis of covolume schemes in general. It also appears in the unstructured Delaunay–Voronoi construction [8,10].

The T nodes of the dual mesh cells are assumed to be numbered sequentially, and likewise the E edges with E' interior edges and the N interior nodes of the primal mesh. The individual primal cells and edges are denoted by τ_i and σ_j respectively. Those of the dual mesh are denoted by primed quantities such as σ'_j. A direction is assigned to each primal edge by the rule that the positive direction is from low to high node number. The dual edges are directed by the corresponding rule.

To begin, the div equation (1.1) is integrated over a primal mesh square τ_1 and the curl equation (1.2) over a dual mesh square τ'_1 using the appropriate Green's theorem in each case, see Figure 1. Hence

$$\sum_{\sigma_i \in \partial\tau'} \int_{\sigma_i} \mathbf{u} \cdot \mathbf{n} \, ds = \int_{\tau} f \, d\mathbf{x}, \tag{2.1}$$

$$\sum_{\sigma'_i \in \partial\tau'} \int_{\sigma'_i} \mathbf{u} \cdot \mathbf{t} \, ds = \int_{\tau'} g \, d\mathbf{x}, \tag{2.2}$$

where $\mathbf{t}$ denotes the unit tangent vector along the co-edge σ'_i and $\mathbf{n}$ is the unit normal vector of the primal edge σ_i.

First, let us introduce a low-order covolume scheme and its properties.

In a low-order scheme, the mid-point rule is used to approximate the line integral along the mesh edges. Referring to Figure 1, (2.1) is approximated by

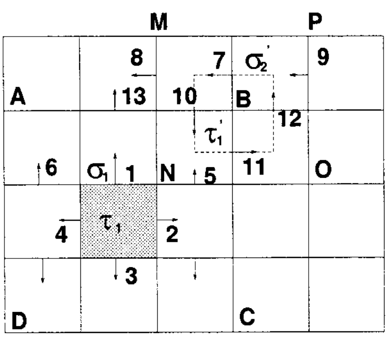

Figure 1. Mesh indicating τ_1 and τ'_1 primal mesh squares.

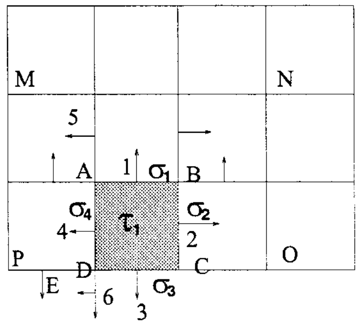

Figure 2. Close up of the primal mesh square τ_1, indicating all parameters.

$$h(u_1 + u_2 + u_3 + u_4) = \int_{\tau_1} f \, d\mathbf{x}. \tag{2.3}$$

Here and below, u_j denotes an approximation to $\mathbf{u}\cdot\mathbf{n}_j$, and $\mathbf{n}_j$ denote unit normals of σ_j. There will be a similar equation for each one of the T primal mesh squares in the grid. In matrix form, with u denoting the vector of components u_j, these *flux equations* can be written as

$$F_1 u = \bar{\rho}, \tag{2.4}$$

where $u \in R^E$ and $\bar{\rho} \in R^T$.

Similarly, (2.2) is approximated by

$$h(u_7 + u_{10} + u_{11} + u_{12}) = \int_{\tau'_1} g \, d\mathbf{x}. \tag{2.5}$$

Assembling these N *circulation equations* gives the matrix equations

$$C_1 u = \bar{\omega}, \tag{2.6}$$

where $u \in R^E$ and $\bar{\omega} \in R^N$.

The boundary condition (1.3) is discretized by (see Figure 2)

$$u_3 = \frac{1}{h} \int_{\sigma_3} l \, ds. \tag{2.7}$$

There are N_1 of these boundary equations where N_1 denotes the number of boundary edges, which is also the number of boundary nodes. Thus (2.3), (2.5) and (2.7) form a linear system of $T + N + N_1$ equations in E unknowns. According to the classical Euler formula

$$T + N + N_1 = E + 1, \tag{2.8}$$

and there is one more equation than unknowns. This turns out to be consistent with Equation (1.4) [8].

Denote the inner product space consisting of R^E equipped with $[\,.\,,\,.\,]$ by U, where

$$[u, v] := \sum_{i=1}^{E} h^2 u_i v_i \tag{2.9}$$

and

$$\|u\|^2 := [u, u].$$

Then we can refer to 'grid functions' $u \in U(\bar{\Omega})$ and regard them as having boundary values $u|_\Gamma$ and interior values $u|_\Omega$. Define

$$U_0 := \{u \in U;\ u|_\Gamma = 0\}.$$

The following theorem is important in the analysis of the covolume scheme and is proved in [8].

Theorem 2.1
If $u, v \in U_0$ and

$$F_1 u = 0, \qquad C_1 v = 0,$$

then

$$[u, v] = 0. \quad \square \tag{2.10}$$

From this theorem, the linear system (2.4), (2.6) and (2.7) can be proved to have a unique solution. Theorem 2.1 is a discrete analog of the following formula in vector calculus:

$$\int_\Omega \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} = 0,$$

if $\mathbf{u}, \mathbf{v} \in H_0(\text{div}; \Omega) \cap H(\text{curl}; \Omega)$ and div $\mathbf{u} = 0$, curl $\mathbf{u} = 0$.

An error estimate for this scheme is proved in [8] and the convergence is second-order for the uniform grid.

3. A HIGHER-ORDER SCHEME

The higher-order covolume scheme uses the following quadrature rule to discretize the line integrals in (2.1) and (2.2):

$$\frac{1}{h}\int_{-h/2}^{h/2} f(x)\, \mathrm{d}x \approx \frac{1}{24}(f(-h) + 22f(0) + f(h)). \tag{3.1}$$

This rule uses data from outside of the integration interval in addition to the mid-point value. It is exact for cubic polynomials and its error is $O(h^4)$, instead of $O(h^2)$ for the mid-point rule in the low-order scheme. This is used to discretize each line integral in (21) and (2.2). For example, for the interior primal edge σ_1, both of whose endpoints are inside Ω, $(1/h)\int_{\sigma_1} \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}s$ is approximated by (see Figure 1)

$$I_{\sigma_1} u := \frac{1}{24}(u_5 + 22u_1 + u_6), \tag{3.2}$$

where u_i denotes an approximation to $\mathbf{u} \cdot \mathbf{n}_i$. Also, for the integrals along the co-edges the same rule is applied. For example, for the co-edge σ'_2 $(1/h)\int_{\sigma'_2} \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}s$ is approximated by (see Figure 1)

$$I_{\sigma'_2} u := \frac{1}{24}(u_8 + 22u_7 + u_9). \tag{3.3}$$

Implementing the boundary conditions for higher-order schemes must be done carefully if the full high order of accuracy is to be obtained. It is apparent that the circulation around the dual mesh squares can be computed without additional effort. However, to compute the flux out of the boundary mesh squares, for example, τ_1 in Figure 2, requires data not in our possession. One reasonable solution is to use the boundary derivatives to design a suitable quadrature rule. For example, to compute $\int_{\sigma_4} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s$ in Figure 2, a point a half-cell below the boundary point D is introduced and (3.1) is used

$$\frac{1}{h}\int_{\sigma_4} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = \frac{1}{24}(22\mathbf{u}\cdot\mathbf{n}_4 + \mathbf{u}\cdot\mathbf{n}_5 + \mathbf{u}\cdot\mathbf{n}_6) + O(h^4). \tag{3.4}$$

To compute $\mathbf{u}\cdot\mathbf{n}_6$, the following expansion is used:

$$\frac{\mathbf{u}\cdot\mathbf{n}_4 - \mathbf{u}\cdot\mathbf{n}_6}{h} = \mathbf{D}_y(\mathbf{u}\cdot\mathbf{t})(D) + \frac{h^2}{24}\mathbf{D}_{yyy}(\mathbf{u}\cdot\mathbf{t})(D) + O(h^4), \tag{3.5}$$

where $\mathbf{D}_y$ and $\mathbf{D}_{yyy}$ denote partial derivatives. So

$$\frac{1}{h}\int_{\sigma_4} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = \frac{1}{24}(23\mathbf{u}\cdot\mathbf{n}_4 + \mathbf{u}\cdot\mathbf{n}_5) - \frac{h}{24}\mathbf{D}_y(\mathbf{u}\cdot\mathbf{t})(D) - \frac{h^3}{24^2}\mathbf{D}_{yyy}(\mathbf{u}\cdot\mathbf{t})(D) + O(h^4). \tag{3.6}$$

Using the curl equation (1.2) one obtains

$$\mathbf{D}_y(\mathbf{u}\cdot\mathbf{t})(D) = \mathbf{D}_x(\mathbf{u}\cdot\mathbf{n}) - g(D) = \mathbf{D}_x l(D) - g(D).$$

To approximate $\mathbf{D}_{yyy}(\mathbf{u}\cdot\mathbf{t})(D)$, u is eliminated from the div–curl system $u_x + v_y = f$, $v_x - u_y = g$, to obtain $v_{xx} + v_{yy} = g_x + f_y$, so that

$$\mathbf{D}_{yyy}(\mathbf{u}\cdot\mathbf{t})(D) = u_{yyy}(D) = v_{xxx}(D) - g_{yyy}(D) = g_{xx}(D) + f_{xy}(D) - g_{yy}(D) - l_{xxx}(D).$$

These derivatives of the boundary data are then approximated by standard finite difference schemes. For example

$$g_{yy}(D) = \frac{4}{h^2}(g(D) - 2g(A') + g(A)) + O(h),$$

where A' is the mid-point of the primal edge σ_4 (see Figure 2) and the approximation is exact if g is a quadratic polynomial.

Denote the finite difference approximations to the derivatives of the boundary data by $\tilde{L}$. After putting $\tilde{L}$ into (3.6) a quadrature rule that is exact for cubic polynomials is obtained. Then

$$I_{\sigma_4}u := \frac{1}{24}(23u_4 + u_5) - \frac{h}{24}(l_x(D) - g(D)) + \tilde{L} \tag{3.7}$$

is used to discretize $(1/h)\int_{\sigma_4} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s$ and the global accuracy is $O(h^4)$.

Putting the flux equations (3.2) and (3.7) together gives

$$Fu = \rho \tag{3.8}$$

where $u \in R^E$ denotes the vector whose ith component is u_i, the approximation to $\mathbf{u}\cdot\mathbf{n}_i$, and $\rho \in R^T$.

Similarly, assembling all the circulation equations (3.3), we have

$$Cu = \omega$$

where $\omega \in R^N$ and N is the number of the interior nodes in the grid. As in the lower-order scheme in Section 2, (3.8), (3.9) and the boundary condition (2.7) form a system of linear equations. There is one more equation than there are unknowns.

Theorem 3.1
The linear system (3.8) and (3.9) has a unique solution.

Proof
There is need only to prove that the homogeneous equations $Fu = 0$, $Cu = 0$, with $u \in U_0$, have a unique solution of 0.

Let $I_\sigma u \in R^E$ denote the vector whose ith component is $I_{\sigma_i} u$, the higher-order quadrature approximation for $(1/h)\int_{\sigma_i} \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}s$, and let $I_{\sigma'} u \in R^E$ denote the vector whose jth component is $I_{\sigma_j'} u$, the higher-order quadrature approximation for $(1/h)\int_{\sigma_j'} \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}s$. Then, by regrouping the terms corresponding to the same primal or dual edges, we obtain

$$F_1(I_\sigma u) = Fu = 0,$$

and

$$C_1(I_{\sigma'} u) = Cu = 0. \quad \square$$

From Theorem 2.1, we have

$$[I_\sigma u, I_{\sigma'} u] = 0. \tag{3.10}$$

Assume for the moment the following lemma:

Lemma 3.1
If $u \in U_0$,

$$\frac{1}{2}\|u\|^2 \le [I_\sigma u, I_{\sigma' u}]. \tag{3.11}$$

Then $\|u\|^2 = 0$ from (3.10), (3.11) and $u = 0$.

Proof

$$[I_\sigma u, I_{\sigma'} u] = h^2 \sum_{i=1}^{E} (I_{\sigma_i} u)(I_{\sigma_i'} u). \tag{3.12}$$

The method of proving the estimate (3.12) is to look at the dominant terms. For example for the interior primal and dual edges σ_1 and σ_1' (see Figure 1) we have

$$(I_{\sigma_1} u)(I_{\sigma_i'} u) = \frac{1}{24^2}(22u_1 + u_5 + u_6)(22u_1 + u_3 + u_{13})$$

$$= \frac{1}{24^2}[22^2 u_1^2 + 44u_1(u_3 + u_{13} + u_5 + u_6) + (u_5 + u_6)(u_3 + u_{13})].$$

Using $-\frac{1}{2}(a^2 + b^2) \le ab$, we find

$$(I_{\sigma_1} u)(I_{\sigma_1'} u) \ge \frac{1}{24^2}[(22^2 - 88)u_1^2 - 22(u_3^2 + u_{13}^2 + u_5^2 + u_6^2) - (u_5^2 + u_6^2 + u_3^2 + u_{13}^2)].$$

Computing the contributions from four surrounding four primal edges σ_5, σ_6, σ_{13}, σ_3 and the corresponding dual edges, it is found that the coefficient of u_1^2 in the right-hand-side of (3.12) is at least

$$\frac{1}{24^2}(22^2 - 88 - 88 - 4) \ge \frac{1}{2}.$$

When σ_1 is a boundary edge a similar estimate is obtained, so

$$
\begin{aligned}
[I_\sigma u, I_{\sigma'} u] &\geq \frac{1}{2} h^2 \sum_{i=1}^{E-N_1} u_i^2 \\
&= \frac{1}{2} h^2 \sum_{i=1}^{E} u_i^2, \quad \text{for } u \in U_0 \\
&= \frac{1}{2} \|u\|^2. \qquad \square
\end{aligned}
$$

Moreover, by direct computation, a similar result to Theorem 2.1 can be proved, i.e.

Lemma 3.2

$$FC^T = 0. \qquad \square \tag{3.13}$$

This identity provides a discrete analog of div(curl $\mathbf{u}$) = 0. Equation (3.13) can be used to give an alternative proof for Theorem 3.1. □

4. ERROR ESTIMATES

In this section, the error in approximating the solution $\mathbf{u}$ of the div–curl system (1.1)–(1.3) will be estimated by the solution u of the covolume approximation (3.8) and (3.9).

To begin, recall the 'mesh functions' introduced in [8]. Let $u^{(1)}$ denote the vector in R^E whose kth component $u_k^{(1)}$ is defined by

$$u_k^{(1)} := \frac{1}{h} \int_{\sigma_k} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s \quad \text{for } k = 1, \ldots, E. \tag{4.1}$$

Similarly, let $u^{(2)}$ denote the vector in $R^{E'}$ whose kth component $u_k^{(2)}$ is

$$u_k^{(2)} := \frac{1}{h} \int_{\sigma_k'} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s \quad \text{for } k = 1, \ldots, E'. \tag{4.2}$$

Finally, let $u^{(3)}$ denote the vector in $R^{E'}$ whose kth component is defined as

$$u_k^{(3)} := \mathbf{u}(P_k) \cdot \mathbf{n}_k \quad \text{for } k = 1, \ldots, E', \tag{4.3}$$

where P_k is the mid-point on the edge σ_k and $\mathbf{n}_k$ is the corresponding unit normal. Since higher-order schemes are considered, it is assumed that the solution of (1.1)–(1.3) is regular enough so that $\mathbf{u}(P_k)$ in (4.3) is well-defined.

The main estimate is the following:

Theorem 4.1

Let $\mathbf{u} \in (H^4(\Omega))^2$ denote the solution of the div–curl system (1.1)–(1.3), and u the solution of the covolume scheme (3.8)–(3.9), then

$$\|u - u^{(3)}\| \leq Kh^4 |\mathbf{u}|_{(H^4(\Omega))^2}. \tag{4.4}$$

First, let us estimate the operators I_σ, $I_{\sigma'}$. □

Lemma 4.1
For any $w \in R^E$, we have

$$\|I_\sigma w\| \le \sqrt{3}\|w\|, \tag{4.5}$$

$$\|I_{\sigma'} w\| \le \sqrt{3}\|w\|. \tag{4.6}$$

Proof

$$\|I_\sigma w\|^2 = \sum_{i=0}^{E} h^2 (I_{\sigma_i} w)^2 = \sum_{i=0}^{E} h^2 \left[\frac{1}{24}(22w_i + w_{i-1} + w_{i+1})\right]^2$$
$$\le \sum_{i=0}^{E} \frac{h^2}{24^2} 3(22^2 w_i^2 + w_{i-1}^2 + w_{i+1}^2) \le 3 \sum_{i=0}^{E} h^2 w_i^2 = 3\|w\|^2.$$

(4.6) is proved similarly. □

Proof of Theorem 4.1
Denote $\tilde{u}^{(1)}$ the vector in R^E such that

$$\tilde{u}_k^{(1)} = u_k^{(1)},$$

if σ_k is an interior primal edge. When σ_k is a boundary primal edge, the difference between $\tilde{u}_k^{(1)}$ and $u_k^{(1)}$ is the boundary derivative terms in (3.7).

First, (3.8) and (3.9) are rewritten as

$$Fu = F_1(I_\sigma u) = \rho = F_1(\tilde{u}^{(1)}), \tag{4.7}$$

$$Cu = C_1(I_{\sigma'} u) = \omega = C_1(u^{(2)}). \tag{4.8}$$

So

$$F_1(I_\sigma u - \tilde{u}^{(1)}) = 0, \qquad C_1(I_{\sigma'} u - u^{(2)}) = 0.$$

From Theorem 2.1, we obtain

$$[I_\sigma u - \tilde{u}^{(1)}, I_{\sigma'} u - u^{(2)}] = 0. \tag{4.9}$$

So it follows that

$$[I_\sigma(u - u^{(3)} + I_\sigma u^{(3)} - \tilde{u}^{(1)}, I_{\sigma'}(u - u^{(3)}) + I_{\sigma'} u^{(3)} - u^{(2)})] = 0.$$

$$\begin{aligned}
[I_\sigma(u - u^{(3)}), I_{\sigma'}(u - u^{(3)})] = &- [I_\sigma(u - u^{(3)}), I_{\sigma'} u^{(3)} - u^{(2)}] - [I_\sigma u^{(3)} - \tilde{u}^{(1)}, I_{\sigma'}(u - u^{(3)})] \\
&- [I_\sigma u^{(3)} - \tilde{u}^{(1)}, I_{\sigma'} u^{(3)} - u^{(2)}] \\
\le &\|I_\sigma(u - u^{(3)})\| \|I_{\sigma'} u^{(3)} - u^{(2)}\| + \|I_\sigma u^{(3)} - \tilde{u}^{(1)}\| \|I_{\sigma'}(u - u^{(3)})\| \\
&+ \|I_\sigma u^{(3)} - \tilde{u}^{(1)}\| \|I_{\sigma'} u^{(3)} - u^{(2)}\| \\
\le &K_1 \|u - u^{(3)}\| (\|I_{\sigma'} u^{(3)} - u^{(2)}\| + \|I_\sigma u^{(3)} - \tilde{u}^{(1)}\|) \\
&+ \|I_\sigma u^{(3)} - \tilde{u}^{(1)}\| \|I_{\sigma'} u^{(3)} - u^{(2)}\|.
\end{aligned}$$

The last step follows from (4.5) and (4.6).

By the coercivity result (3.11) in Lemma 3.1, we obtain

$$\frac{1}{2}\|u-u^{(3)}\|^2 \le K_1\|u-u^{(3)}\|(\|I_{\sigma'}u^{(3)}-u^{(2)}\| + \|I_\sigma u^{(3)}-\tilde{u}^{(1)}\|) + \|I_\sigma u^{(3)}-\tilde{u}^{(1)}\|\|I_{\sigma'}u^{(3)}-u^{(2)}\|. \tag{4.10}$$

To estimate $I_{\sigma'}u^{(3)}-u^{(2)}$ and $I_\sigma u^{(3)}-\tilde{u}^{(1)}$, its components are examined. For the dual edge σ'_2 of the dual cell τ'_1 (see Figure 1), we have

$$(I_{\sigma'}u^{(3)}-u^{(2)})_2 = \frac{1}{24}(22\mathbf{u}_7\cdot\mathbf{n}_7\cdot\mathbf{n}_8\cdot\mathbf{u}_9\cdot\mathbf{n}_9) - \frac{1}{h}\int_{\sigma'_2}\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s.$$

$(I_{\sigma'}u^{(3)}-u^{(2)})_2$ is a linear continuous functional on $(H^4(\Omega))^2$ (by a Sobolev embedding theorem) and, from (3.3), it vanishes for cubic polynomial vectors $\mathbf{u}$. So

$$\left|(I_{\sigma'}u^{(3)}-u^{(2)})_2\right| \le K(\tilde{\tau})|\mathbf{u}|_{(H^4(\tilde{\tau}))^2}, \tag{4.11}$$

where $\tilde{\tau}$ is the rectangular region MPON, the union of the square cells that contain the dual edge σ'_2. A scale change argument shows that $K(\tilde{\tau})$ depends on h^3 [8], $K(\tilde{\tau}) = K_1h^3$, where K_1 is independent of h. So

$$\begin{aligned}\|I_{\sigma'}u^{(3)}-u^{(2)}\|^2 &= \sum_{i=1}^{E-N_1}(I_\sigma u^{(3)}-u^{(2)})_i^2h^2 \le \sum_{i=1}^{E-N_1}(K_1h^3|\mathbf{u}|_{(H^4(\tilde{\tau}_i))^2})^2h^2 \le K_1^2h^8\sum_{i=1}^{E-N_1}|\mathbf{u}|^2_{(H^4(\tilde{\tau}_i))^2}\\ &\le K_2h^8|\mathbf{u}|^2_{(H^4(\Omega))^2},\end{aligned}$$

where the last inequality follows from the fact that the $\tilde{\tau}_i$ overlap one other only four times. Thus,

$$\|I_{\sigma'}u^{(3)}-u^{(2)}\| \le K_3h^4|\mathbf{u}|_{(H^4(\Omega))^2}. \tag{4.12}$$

Similarly, from (3.2) and (3.8) it can be shown that

$$\|I_\sigma u^{(3)}-\tilde{u}^{(1)}\| \le K_3h^4|\mathbf{u}|_{(H^4(\Omega))^2}. \tag{4.13}$$

Putting (4.12) and (4.13) into (4.10), we obtain

$$\|u-u^{(3)}\|^{(2)} \le K_3h^4|\mathbf{u}|_{(H^4(\Omega))^2}\|u-u^{(3)}\| + K_4h^8|\mathbf{u}|^2_{(H^4(\Omega))^2},$$

so that

$$\left(\|u-u^{(3)}\| - \frac{K_3}{2}h^4|\mathbf{u}|_{(H^4(\Omega))^2}\right)^2 \le \left(K_4+\frac{K_3^2}{2}\right)h^8|\mathbf{u}|^2_{(H^4(\Omega))^2}$$

and

$$\|u-u^{(3)}\| \le \left(\frac{K_3}{2}+\sqrt{K_4+\frac{k_3^2}{2}}\right)h^4|\mathbf{u}|_{(H^4(\Omega))^2},$$

and the estimate (4.4) is proved. □

A numerical example has been computed. The computational domain is a unit square $[0, 1]\times[0, 1]$. The domain was divided into equal small squares with dimension $h\times h$. Then a dual mesh was generated (dual squares) by connecting the circumcenters of any adjacent primal square cells. The following problem is considered

$$\operatorname{div}\mathbf{u} = 0,$$

$$\operatorname{curl}\mathbf{u} = \omega,$$

Table I. Error between u and $u^{(3)}$

h	0.2	0.1	0.005
$\|u-u^{(3)}\|$	3.29d−2	2.038d−3	1.269d−4

$$u|_{x=0} = 0,$$
$$u|_{x=1} = \sin(10)\cos(10y),$$
$$v|_{y=0} = 0,$$
$$v|_{y=1} = -\sin(10)\cos(10x),$$

where $\mathbf{u} = (u, v)$ and $\omega = 20\sin(10x)\sin(10y)$.

The exact solution of this problem is

$$\mathbf{u} = \begin{pmatrix} \sin(10x)\cos(10y) \\ -\cos(10x)\sin(10y) \end{pmatrix}.$$

Three meshes were used in the computation, $h = 0.2$ for the coarse mesh and $h = 0.05$ for the fine mesh. The results are shown in Table I. The average rate for this example is about $h^{4.00994}$, which is almost the same as the rate given by Theorem 4.1.

REFERENCES

1. R.A. Nicolaides, 'Flow discretization by complementary volume techniques', *AIAA paper 89-1979*, *Proceeding of 9th AIAA CFD Meeting*, Buffalo, NY, June 1989.
2. S. Choudhury and R.A. Nicolaides, 'Discretization of incompressible vorticity–velocity equations on triangular meshes', *Int. J. Numer. Methods Fluid Dyn.*, **11**, 221 (1990).
3. B.J. McCartin, 'Discretization of the semiconductor device equations', in J.J.H. Miller (ed.), *New Problems and Solutions for Device and Process Modeling (NASECODE IV)*, Boole Press, Dublin, 1986.
4. B.J. McCartin and J.F. Dicello, 'Three-dimensional finite difference frequency domain scattering computation using the control region approximation', *IEEE Trans. Magnet.*, **25**, 3092 (1989).
5. E.G. Puckett, 'Vortex methods: an introduction and survey of selected research topics', in M.D. Gunzburger and R.A. Nicolaides (eds.), *Incompressible Fluid Dynamics*, Cambridge University Press, Cambridge, 1993, p. 335.
6. C.-L. Chang and M.D. Gunzburger, 'Finite element methods for first-order elliptic systems in three dimensions', *Appl. Math. Comp.*, **23**, 171 (1987).
7. S. Wong and Z. Cendes, 'Combined finite element–modal solution of three-dimensional eddy current problems', *IEEE Trans. Magnet.*, **24**, 2685 (1988).
8. R.A. Nicolaides, 'Direct discretizations of planar div–curl problems', *SIAM J. Numer. Anal.*, **29**, 32–56 (1992).
9. V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer, Berlin, 1986.
10. R.A. Nicolaides and X. Wu, 'Covolume solutions of three-dimensional div–curl equations', *SIAM J. Numerical Anal.*, **34**, 2195 (1977).

